STRESSES IN A PLATE KEPT IN EQUILIBRIUM BY A PIN

V. N. Solodovnikov

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The state of strain in plates connected by bolts or pins has several times been examined. For example, infinite elastic orthotropic plates with circular holes have been examined [1], with the plates loaded by pins in the holes. The solution is defined for two contacting elastic bodies (plate and pin) as complex Fourier series. The effects of the pin elasticity on the stresses in the plate are minor. Studies have also been made on the effects of the plate orthotropy parameters, the friction, and the gap between the pin and the edge of the plate. In all these cases, the contact region occupies less than half of the edge of the hole, and the maximal circumferential stress occurs in the part where there is no contact near the point where the plate is close to the pin.

The stresses around a single bolt or pin in a finite plate have been calculated by the finite-element method [2-4]. The stress distribution has been derived [2] in a layered composite around a hole loaded by an elastic pin. The contact zone was determined. The radial pressure on the pin differed from sinusoidal. In [3, 4], the pin was taken as absolutely rigid, and the boundary conditions at the edge of the hole were not defined but instead set arbitrarily. It was assumed that the contact region occupied half of the edge. In [3], a cosine radial pressure distribution was specified there, while in [4], zero radial displacements were defined in accordance with a mode of loading different from that in [3]. The [5, 6] experiments in general confirm the stress distribution in the plate around a pin as found theoretically in [1], but the boundaries of the contact region were not given. There may be nonzero radial stresses at the edge of the hole only in the contact region, but in [5] they were observed for half of the edge and in [6] for more than half of it.

Here I consider a circular plate for convenience, which is loaded in a way of practical interest. At the outer edge of the plate, the conditions correspond to one-sided uniaxial loading. The pin is taken as immobile. A treatment of Signorini type is formulated having an unknown contact region. Finite-element methods are used to derive the state of stress and strain.

1. Basic Equations. We consider the planar stress states in a polar coordinate system. We express the strains in terms of the displacements, the equilibrium equation, and Hooke's law as [7]

$$e_{11}=u_{1,1},\ e_{22}=\frac{1}{r}(u_{2,2}+u_1),\ e_{12}=\frac{1}{2r}(u_{1,2}+ru_{2,1}-u_2),$$
$$(r\sigma_{11})_{,1}+\sigma_{12,2}-\sigma_{22}+rp_1=0,$$
$$\sigma_{22,2}+(r\sigma_{12})_{,1}+\sigma_{12}+rp_2=0,$$
$$e_{11}=\frac{1}{E}(\sigma_{11}-\nu\sigma_{22}),\ \ e_{22}=\frac{1}{E}(\sigma_{22}-\nu\sigma_{11}),\ \ e_{12}=\frac{1+\nu}{E}\sigma_{12}.$$

Here $u_i$ are the displacements, $e_{ik}$ are the strains, $\sigma_{ik}$ are the stresses, and $p_i$ the bulk forces (i, k = 1, 2) in the polar coordinate system (r, θ), with r radius, θ angle, E Young's modulus, and ν Poisson's ratio. The subscripts 1 and 2 denote quantities referred to the r and θ axes, while 1 and 2 after commas denote partial differentiation with respect to r and θ correspondingly. The intensity of the tangential stresses is

$$J=\left[\frac{1}{3}(\sigma_{11}^2-\sigma_{11}\sigma_{22}+\sigma_{22}^2)+\sigma_{12}^2\right]^{1/2}.$$

We introduce the functional

$$\Phi=\iint\limits_{\Omega}\Big\{\frac{E}{2(1-\nu^2)}\left[e_{11}^2+2\nu e_{11}e_{22}+e_{22}^2+2(1-\nu)e_{12}^2\right]-$$

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$$- p_1u_1 - p_2u_2\Big\} r\,dr\,d\theta - \int_{\Gamma} [\sigma_{nn}(u_1l_2 - u_2l_1) + \sigma_{nl}(u_1l_1 + u_2l_2)]\,dl \qquad (1.1)$$

in the region $\Omega$ occupied by the body and bounded by the edge $\Gamma$. The integration over $\Gamma$ is done with $\Omega$ to the left of $\Gamma$; $\sigma_{nn}$ and $\sigma_{n\ell}$ are the normal and tangential stresses, which are defined by

$$\sigma_{nn} = \sigma_{11}l_2^2 - 2\sigma_{12}l_1l_2 + \sigma_{22}l_1^2,$$
$$\sigma_{nl} = (\sigma_{11} - \sigma_{22})l_1l_2 + \sigma_{12}(l_2^2 - l_1^2),\ l_1 = dr/dl,$$
$$l_2 = r d\theta/dl,\ dl = [(dr)^2 + (rd\theta)^2]^{1/2}$$

($\ell_1$ and $\ell_2$ are the components of the unit vector tangential to $\Gamma$).

Here the deformations should be expressed in terms of the displacements. The displacements are varied. The values of $\sigma_n$ and $\sigma_{n\ell}$ in the integral over $\Gamma$ are fixed on varying $\Phi$.

As $\Phi$ is stationary, one gets equilibrium equations and expressions for the normal and tangential stresses on $\Gamma$ in terms of the displacements. With given boundary conditions, those equations give the displacements, which themselves give all the other functions.

2. Formulation. We have an annular elastic isotropic plate having ratio of outside radius to inside radius $R/\rho = 5$ (Fig. 1). Within the hole is a pin with the same radius $\rho$. On the right-hand half of the outer edge of the plate, conditions are specified for uniaxial loading along the x axis with stresses $\sigma_{xx} = S$ (the arrows in Fig. 1 along the x axis show the loading vectors), while the left-hand half edge is not loaded. There are no bulk forces ($p_1 = p_2 = 0$). The pin is not displaced on loading. We neglect any deformation in the pin and the friction between it and the edge of the plate.

The system is symmetrical about the x axis, so we consider only the upper half of the plate (Fig. 1) with the following boundary conditions:

$$\sigma_{11} = S\cos^2\theta,\ \sigma_{12} = -S\sin\theta\cos\theta \text{ for } r = R,\ 0 \leqslant \theta \leqslant 0.5\pi; \qquad (2.1)$$
$$\sigma_{11} = \sigma_{12} = 0 \text{ for } r = R,\ 0.5\pi \leqslant \theta \leqslant \pi;$$
$$u_2 = 0,\ \sigma_{12} = 0 \text{ for } \rho \leqslant r \leqslant R,\ \theta = 0 \text{ and } \theta = \pi; \qquad (2.2)$$
$$u_1 = 0,\ \sigma_{12} = 0,\ \sigma_{11} \leqslant 0 \text{ for } \sigma_{11} = \sigma_{12} = 0,\ u_1 \geqslant 0 \text{ for } r = \rho,\ 0 \leqslant \theta \leqslant \pi. \qquad (2.3)$$

The radial displacements are $u_1 = 0$ at the points of contact between the plate and pin in (2.3), with the edge of the plate pressed onto the pin, so the radial stresses $\sigma_{11}$ cannot be tensile ($\sigma_{11} \leqslant 0$). At points where there is no contact, the edge of the plate is unloaded ($\sigma_{11} = \sigma_{12} = 0$), and the distance from the edge of the plate to the center of the pin should be not less than the radius of the pin, i.e., $u_1^2 + u_2^2 + 2\rho u_1 \geqslant 0$, which on linearization is replaced by $u_1 \geqslant 0$. We have a system of Signorini type [8] having an unknown region of contact between plate and pin. Defining that region is facilitated by the simple geometry. One naturally assumes that the region occupies a certain segment $r = \rho$, $\theta_c \leqslant \theta \leqslant \pi$, and then we get boundary conditions at the internal edge of the plate:

$$\sigma_{11} = \sigma_{12} = 0 \text{ for } r = \rho,\ 0 \leqslant \theta \leqslant \theta_c; \qquad (2.4)$$
$$u_1 = 0,\ \sigma_{12} = 0 \text{ for } r = \rho,\ \theta_c \leqslant \theta \leqslant \pi.$$

At $r = \rho$ and $\theta = \theta_c$, $\sigma_{11} = 0$.

3. Finite-Element Treatment with a Fixed Contact Area. Let the contact between the plate and pin be known. Then the finite-element treatment for the equilibrium with boundary conditions $\Omega$ is as follows. The region $\Omega$ in which we seek the solution is split up into rectangular nine-node isoparametric elements [9] in which the displacements in an element are approximated by quadratics in each of the variables r and $\theta$. The finite-element equations are formulated in terms of the node displacements on the basis of the minimum potential energy principle of (1.1). The integrals over the area of an element and over the sides on $\Gamma$ are based on a three-point Gauss quadrature formula [9]. The conditions for continuity in the displacements apply at the boundaries between elements. At each node, the sum of all the forces acting there is set as zero as defined from the virtual-displacement principle [9].

The complete system is solved by compact Gauss exclusion on the basis of the global rigidity matrix being symmetrical and of strip type [9-11].

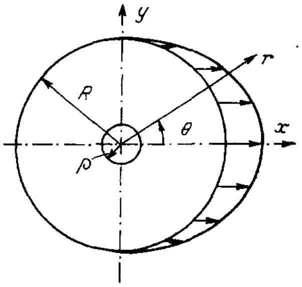


Fig. 1

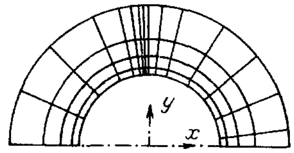


Fig. 2

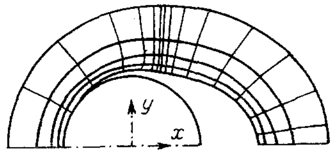


Fig. 3

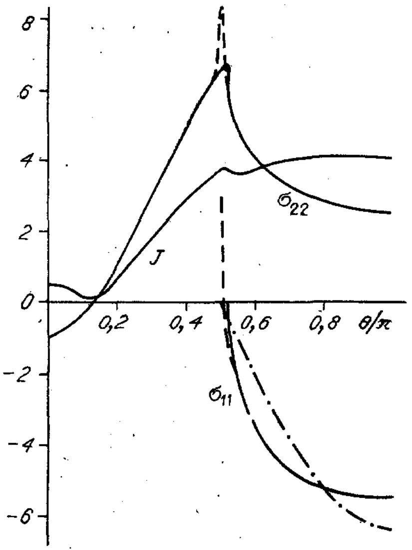


Fig. 4

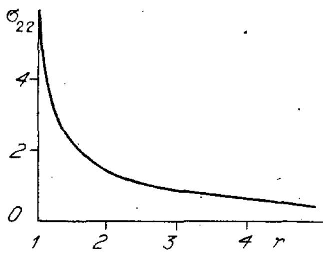


Fig. 5

4. Results. The angular coordinate $\theta_c$ for the extreme point in the contact region is found by dividing the segment into halves as the root for the radial stress $\sigma_{11}$ calculated for $r = \rho$ and $\theta = \theta_c$ from the solution for the equilibrium subject to the boundary conditions (2.1), (2.2), (2.4) and may be considered as a function of $\theta_c$. In our solution, we successively tested $\theta_c/\pi = 0.5$, 0.52, 0.525, 0.5225, and we found that $\sigma_{11} = 0$ for $r = \rho$ and $\theta = \theta_c$ with the required accuracy if $\theta_c = 0.5225\pi$, and then conditions (2.3) are satisfied.

We now describe the equilibrium subject to (2.1), (2.2), (2.4) when $\theta_c = 0.5225\pi$. Poisson's ratio is $\nu = 0.3$.

Then the text and the figures give the dimensionless quantities

$$r' = \frac{1}{\rho}r, \ \left(\sigma_{11}', \sigma_{22}', \sigma_{12}', J'\right) = \frac{1}{S}(\sigma_{11}, \sigma_{22}, \sigma_{12}, J),$$

where a prime to a dimensionless quantity is for brevity omitted.

The plate is split up into quadrilateral nine-mode isoparametric Lagrangian elements by two facilities of coordinate lines: 1) radial lines beginning at $\theta = 0$ with intervals in $\theta$ of $\pi$ multiplied by factors (0.04, 0.08, 0.1, 0.1025, 0.1, 0.05, 0.03, 0.015, 0.01, 0.015, 0.03, 0.06, 0.12, 0.12, 0.1275) and 2) circumferential lines beginning from the edge $r = 1$ with intervals in the radius r of 0.1, 0.15, 0.25, 0.5, 1, 1, 1. Figure 2 shows the decomposition of the inner region $1 \leqslant r \leqslant 2$. The dimensions of the elements are reduced as one approaches the extreme point in the contact region $r = 1$ and $\theta = \theta_c = 0.5225\pi$. In all we obtained 105 elements and 886 unknown variables, the displacement components.

Figure 3 shows the state of strain in the part of the plate shown in Fig. 2. The Cartesian coordinates x and y for any node in the deformed state in Fig. 3 are defined by

$$x = (r + u_1)\cos\theta - u_2\sin\theta, \ y = (r + u_1)\sin\theta + u_2\cos\theta.$$

Here r and $\theta$ are the initial coordinates and one substitutes for $u_1$ and $u_2$ normalized in such a way that the maximum absolute values are all nodes of 1, i.e., they are multiplied by the same positive number for all displacement components. On account of the linearization, the segment of the edge of the hole $\theta_c \leqslant \theta \leqslant \pi$, is represented as the region of contact between the plate and pin and in Fig. 3 does not lie completely on the pin (semicircle with center at the origin).

The solid lines in Fig. 4 show the stresses at the edge of the hole. $\sigma_{11}$ is zero within the accuracy used in the part $\theta \leqslant \theta_c$, where there is no contact between plate and pin, and is negative in the contact part $\theta < \theta_c$; the $\sigma_{11}$ distribution in the contact area differs appreciably from the $(20/\pi)\cos\theta$ one represented in Fig. 4 by the dot–dash line. The circumferential stress $\sigma_{22}$ increases sharply at $\theta < \theta_c$ before the point where the plate contacts the pin at $\theta = \theta_c$. In the contact range $\theta > \theta_c$, $\sigma_{22}$ decreases. The tangential stress intensity J is maximal in the contact part, but it does not have peaks such as are found from the rise in $\sigma_{22}$ at the point of contact between the plate and pin. Throughout the plate, the maximal values $\sigma_{11} = -5.4696$, $\sigma_{22} = 6.7639$, and $J = 4.1073$ are attained at the edge of the hole.

Figure 5 shows $\sigma_{22}$ in the radial section $\theta = \theta_c$.

Specifying a large contact region increases the maximum stresses. The dashed lines in Fig. 4 show the change in $\sigma_{11}$ and $\sigma_{22}$ obtained on solving with (2.1), (2.2), (2.4) if $\theta_c = 0.5\pi$. If the contact region is shorter than it should be, $\sigma_{11}$ at the extreme point takes nonzero negative values.

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